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LETTER TO THE EDITOR

Transformations of solutions for equations and hierarchies of pseudo-spherical type

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Abstract
It is known that if an equation describes non-trivial one-parameter families of pseudo-spherical surfaces, its conservation laws, (generalized, nonlocal) symmetries and Bäcklund transformations can be studied by geometrical means [4, 10]. In this letter it is pointed out that there exist correspondences, or 'generalized Bäcklund transformations', between *arbitrary* solutions (satisfying some genericity conditions) of any two single equations describing pseudo-spherical surfaces. Then, the notion of a hierarchy of equations of pseudo-spherical type is introduced, and a theorem stating that there also exist correspondences between arbitrary solutions of any two such hierarchies is presented. A full account of these results appears elsewhere [12, 13].

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1. Equations of pseudo-spherical type

Equations of pseudo-spherical type were introduced by Chern and Tenenblat [4, 15] motivated by an observation due to Sasaki [14]: generic solutions of equations integrable by the classical Ablowitz, Kaup, Newell and Segur (AKNS) inverse-scattering scheme determine—whenever their associated linear problems are real—pseudo-spherical surfaces.

Below and henceforth, $u_{x^p t^q}$ stands for $\partial^{p+q}u/\partial x^p \partial t^q$, $\alpha = 1, 2, 3$ and $\beta = 1, 2$.

Definition 1. *A scalar differential equation* $\Xi(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0$ *in two independent variables* x, t *is of pseudo-spherical type (or, 'it describes pseudo-spherical surfaces'; or, it is a 'PSS equation') if there exist one-forms* $\omega^\alpha \neq 0$,

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_{x^r t^p})\,\mathrm{d}x + f_{\alpha 2}(x, t, u, \ldots, u_{x^s t^q})\,\mathrm{d}t \tag{1}$$

whose coefficients $f_{\alpha\beta}$ are differential functions (that is, smooth functions depending on x, t, and a finite number of derivatives of u) such that the one-forms $\overline{\omega}^{\alpha} = \omega^{\alpha}(u(x,t))$ satisfy the structure equations

$$\mathrm{d}\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2 \qquad \mathrm{d}\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3 \qquad \mathrm{d}\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2 \tag{2}$$

whenever $u = u(x,t)$ is a solution to $\Xi = 0$.

The trivial case when the functions $f_{\alpha\beta}$ all depend only on x, t is excluded from further consideration. Definition 1 yields constructions of zero-curvature representations, linearizations, Bäcklund transformations, conservation laws and (generalized/nonlocal) symmetries, and it has also been used to compare some of the integrability notions appearing in the literature. Details may be found in [1, 4, 6, 10, 11, 14, 15] and references therein.

Example 1. The Hunter–Saxton equation $u_{xxt} = -u_{xxx}u - 2u_{xx}u_x$ introduced in [7] is of pseudo-spherical type with associated one-forms ω^{α} given by

$$\omega^1 = (u_{xx} - \beta)\,\mathrm{d}x + \left(\frac{u_x - u_x\beta}{\eta} + \frac{1-\beta}{\eta^2} - uu_{xx} - 1 + u\beta\right)\mathrm{d}t \tag{3}$$

$$\omega^2 = \eta\,\mathrm{d}x + \left(\frac{1-\beta}{\eta} - \eta u + u_x\right)\mathrm{d}t \tag{4}$$

$$\omega^3 = (u_{xx} + 1)\,\mathrm{d}x + \left(-uu_{xx} - u + \frac{1-\beta}{\eta^2} + \frac{u_x - u_x\beta}{\eta}\right)\mathrm{d}t \tag{5}$$

in which the parameters η and β are constrained by the relation $\eta^2 + \beta^2 - 1 = 0$.

The interesting Camassa–Holm equation also describes pseudo-spherical surfaces [11], and it is not difficult to show that the same holds for equations admitting quadratic pseudo-potentials [10]. Thus, the equations considered by Nucci [9], for instance, are all of pseudo-spherical type. Complete classifications of equations of pseudo-spherical type have been also achieved (see [4, 8, 10], and references therein).

Definition 2 allows one to equip solutions $u(x,t)$ of the PSS equation $\Xi = 0$ with (pseudo-)Riemannian structures:

Definition 2. *Let $\Xi = 0$ be an equation describing pseudo-spherical surfaces with associated one-forms ω^{α}. A solution $u(x,t)$ of $\Xi = 0$ is I-generic if $(\omega^3 \wedge \omega^2)(u(x,t)) \neq 0$, II-generic if $(\omega^1 \wedge \omega^3)(u(x,t)) \neq 0$ and III-generic if $(\omega^1 \wedge \omega^2)(u(x,t)) \neq 0$.*

For example, the associated one-forms (3)–(5) imply that $u(x,t) = x$ is a *III*-generic solution of the Hunter–Saxton equation.

Proposition 1. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^{α}, let $u(x,t)$ be a solution to $\Xi = 0$ and set $\overline{\omega}^{\alpha} = \omega^{\alpha}(u(x,t))$.*

(a) If $u(x,t)$ is I-generic, $\overline{\omega}^2$ and $\overline{\omega}^3$ determine a Lorentz metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$ with connection one-form $\overline{\omega}^1$.
(b) If $u(x,t)$ is II-generic, $\overline{\omega}^1$ and $-\overline{\omega}^3$ determine a Lorentz metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$ with connection one-form $\overline{\omega}^2$.
(c) If $u(x,t)$ is III-generic, $\overline{\omega}^1$ and $\overline{\omega}^2$ determine a Riemannian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$ with connection one-form $\overline{\omega}^3$.

Proposition 1 is proven by using the Cartan structure equations for (pseudo) Riemannian surfaces, which may be found in [15]. It may be also understood in terms of gauge theory (see [12] and [2]).

2. Correspondence results for PSS equations

An important correspondence between solutions to PSS equations was noted by Kamran and Tenenblat [8] (see also Ding and Tenenblat [5]): motivated by the fact that two Riemannian surfaces of constant Gaussian curvature -1 are *locally indistinguishable*, the authors of [8] showed that given two PSS equations, and *III*-generic solutions $u(x,t)$ and $\widehat{u}(\widehat{x},\widehat{t})$ of them, one can relate $u(x,t)$ and $\widehat{u}(\widehat{x},\widehat{t})$ by integrating first-order equations. Moreover, they obtained a *formula* for $\widehat{u}(\widehat{x},\widehat{t})$ in terms of $u(x,t)$.

This result requires explicit changes of independent variables and it is not restricted to transforming solutions of a same equation, or even of equations of the same order: it goes well beyond classical Bäcklund transformations. Due to its importance, it is natural to try to find other theorems of this type. What allows one to extend the Kamran–Tenenblat result is proposition 1. Since two pseudo-Riemannian surfaces of constant Gaussian curvature -1 are also locally isometric, one expects correspondence theorems for *I*- and *II*-generic solutions, exactly as in the Riemannian case. In the *I*-generic case, for instance, one finds [12]:

Theorem 1. *Let* $\Xi(x,t,u,\ldots)=0$ *and* $\widehat{\Xi}(\widehat{x},\widehat{t},\widehat{u},\ldots)=0$ *be two PSS equations with associated one-forms* $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ *and* $\widehat{\omega}^\alpha = \widehat{f}_{\alpha 1}\,\mathrm{d}\widehat{x} + \widehat{f}_{\alpha 2}\,\mathrm{d}\widehat{t}$*, respectively. For any I-generic solutions* $u(x,t)$ *of* $\Xi=0$ *and* $\widehat{u}(\widehat{x},\widehat{t})$ *of* $\widehat{\Xi}=0$*, there exists a local diffeomorphism* $\Psi : V \to \widehat{V}$*, in which* V *and* $\widehat{V}$ *are open subsets of the domains of* $u(x,t)$ *and* $\widehat{u}(\widehat{x},\widehat{t})$ *respectively, and a smooth function* $\nu : V \to \mathbf{R}$*, such that* $\omega^\alpha(u(x,t))$ *and* $\widehat{\omega}^\alpha(\widehat{u}(\widehat{x},\widehat{t}))$ *satisfy*

$$\Psi^*\widehat{\omega}^1 = \omega^1 + \mathrm{d}\nu \qquad \Psi^*\widehat{\omega}^2 = \omega^2\cosh\nu + \omega^3\sinh\nu$$
$$\Psi^*\widehat{\omega}^3 = \omega^2\sinh\nu + \omega^3\cosh\nu. \tag{6}$$

That the maps Ψ and ν exist is simply an expression of the local uniqueness of surfaces of constant curvature referred to above. Write $\Psi(x,t) = (\gamma(x,t), \delta(x,t))$ and denote by M and $\widehat{M}$ the spaces of the independent variables (x,t) and $(\widehat{x},\widehat{t})$ respectively. A careful analysis of (6) allows one to find a system of equations for γ, δ and ν which can be solved without previous knowledge of $\widehat{u}(\widehat{x},\widehat{t})$:

Lemma 1. *Let* $\Xi(x,t,u,\ldots)=0$ *and* $\widehat{\Xi}(\widehat{x},\widehat{t},\widehat{u},\ldots)=0$ *be two PSS equations with associated one-forms* $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ *and* $\widehat{\omega}^\alpha = \widehat{f}_{\alpha 1}\,\mathrm{d}\widehat{x} + \widehat{f}_{\alpha 2}\,\mathrm{d}\widehat{t}$*, respectively, in which* $\widehat{f}_{11} = \widehat{u}$*. Let* $\Psi(x,t) = (\gamma(x,t),\delta(x,t))$ *be a smooth map with Jacobian* $J = \gamma_x\delta_t - \gamma_t\delta_x$ *from (an open subset of)* M *to (an open subset of)* $\widehat{M}$*, and let* ν *be a smooth map from (an open subset of)* M *to* $\mathbf{R}$*. The system of equations*

$$J(\Psi^*\widehat{f}_{12}) = -[\gamma_t(f_{11}+\nu_x) - \gamma_x(f_{12}+\nu_t)] \tag{7}$$
$$(\Psi^*\widehat{f}_{21})\gamma_x + (\Psi^*\widehat{f}_{22})\delta_x = f_{21}\cosh\nu + f_{31}\sinh\nu \tag{8}$$
$$(\Psi^*\widehat{f}_{21})\gamma_t + (\Psi^*\widehat{f}_{22})\delta_t = f_{22}\cosh\nu + f_{32}\sinh\nu \tag{9}$$
$$(\Psi^*\widehat{f}_{31})\gamma_x + (\Psi^*\widehat{f}_{32})\delta_x = f_{21}\sinh\nu + f_{31}\cosh\nu \tag{10}$$
$$(\Psi^*\widehat{f}_{31})\gamma_t + (\Psi^*\widehat{f}_{32})\delta_t = f_{22}\sinh\nu + f_{32}\cosh\nu \tag{11}$$

in which the pull-backs of $\widehat{u}$ *and its derivatives with respect to* $\widehat{x},\widehat{t}$ *appearing in the functions* $(\Psi^*\widehat{f}_{\alpha\beta})(x,t)$ *have been evaluated by means of the equation*

$$\widehat{u}\circ\Psi = \frac{1}{J}(\delta_t(f_{11}+\nu_x) - \delta_x(f_{12}+\nu_t)) \tag{12}$$

admits—whenever $u(x,t)$ *is a I-generic solution of* $\Xi = 0$*—a local solution* $\gamma(x,t)$*,* $\delta(x,t)$*,* $\nu(x,t)$ *such that* $\Psi(x,t) = (\gamma(x,t),\delta(x,t))$ *is a local diffeomorphism.*

Lemma 1 now yields the following transformation result:

Theorem 2. *Let $\Xi(x, t, u, \ldots) = 0$ and $\widehat{\Xi}(\widehat{x}, \widehat{t}, \widehat{u}, \ldots) = 0$ be two PSS equations with associated one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ and $\widehat{\omega}^\alpha = \widehat{f}_{\alpha 1}\,\mathrm{d}\widehat{x} + \widehat{f}_{\alpha 2}\,\mathrm{d}\widehat{t}$, respectively, and assume that $\widehat{f}_{11} = \widehat{u}$. Any I-generic solution $u(x, t)$ of $\Xi(x, t, u, \ldots) = 0$ gives rise to a I-generic solution $\widehat{u}(\widehat{x}, \widehat{t})$ of $\widehat{\Xi}(\widehat{x}, \widehat{t}, \widehat{u}, \ldots) = 0$ by means of*

$$\widehat{u} \circ \Psi = \frac{1}{J}(\delta_t(f_{11} + \nu_x) - \delta_x(f_{12} + \nu_t)) \tag{13}$$

in which ν is a real-valued function, $\Psi(x, t) = (\gamma(x, t), \delta(x, t))$ is a local diffeomorphism, J is the Jacobian of Ψ, and both ν and Ψ are determined by the solution $u(x, t)$ and equations (7)–(11).

Example 2 (Straightening-out *I*-generic solutions)**.** Consider a PSS equation $\Xi = 0$ with associated one-forms (1). Theorem 2 allows one to obtain a *I*-generic solution of the linear second-order equation $\widehat{u}_{\widehat{t}} = \widehat{u}_{\widehat{x}\widehat{x}} + \widehat{u}_{\widehat{x}}$ from a *I*-generic solution $u(x, t)$ to $\Xi = 0$. Indeed, the equation $\widehat{u}_{\widehat{t}} = \widehat{u}_{\widehat{x}\widehat{x}} + \widehat{u}_{\widehat{x}}$ describes pseudo-spherical surfaces with associated functions $\widehat{f}_{\alpha\beta}$ given by

$$\widehat{f}_{11} = \widehat{u} \qquad \widehat{f}_{12} = \widehat{u}_{\widehat{x}} \qquad \widehat{f}_{21} = -1 \qquad \widehat{f}_{22} = 0 \qquad \widehat{f}_{31} = -\widehat{u} \qquad \widehat{f}_{32} = -\widehat{u}_{\widehat{x}}. \tag{14}$$

Now, define functions $\theta(x, t)$, $\alpha(x, t)$ and $\beta(x, t)$ by means of the equations

$$\begin{aligned} &\omega^1 + \mathrm{d}\theta = \omega^2 \sinh\theta + \omega^3 \cosh\theta \qquad \mathrm{d}(\ln\alpha) = \omega^2 \cosh\theta + \omega^3 \sinh\theta \\ &(1/\alpha)\,\mathrm{d}\beta = \omega^2 \sinh\theta + \omega^3 \cosh\theta. \end{aligned} \tag{15}$$

That these functions exist whenever $u(x, t)$ is a solution to $\Xi = 0$ is guaranteed by the structure equations (2). The map $\Psi(x, t) = (\gamma(x, t), \delta(x, t))$, in which

$$\gamma(x, t) = -\ln\left|\alpha(x, t)\left(\frac{\beta(x, t)^2}{\alpha(x, t)^2} - 1\right)\right|$$

$$\delta(x, t) = 1 - \frac{\beta(x, t)}{\alpha(x, t)} + \ln\left|\alpha(x, t)\left(\frac{\beta(x, t)^2}{\alpha(x, t)^2} - 1\right)\right|$$

is a local diffeomorphism from the space of independent variables (x, t) to the space of independent variables $(\widehat{x}, \widehat{t})$ since $u(x, t)$ is *I*-generic. One can check that $\Psi(x, t)$ and the function $\mu(x, t) = \theta(x, t) - \widehat{\theta} \circ \Psi^{-1}(x, t)$, where

$$\widehat{\theta}(\widehat{x}, \widehat{t}) = \ln\left|\frac{2 - \widehat{x} - \widehat{t}}{\widehat{x} + \widehat{t}}\right|$$

satisfy the system of equations (7)–(11). Formula (13) yields $\widehat{u}(\widehat{x}, \widehat{t}) = \widehat{x} + \widehat{t}$.

3. Symmetries and hierarchies of PSS equations

Motivated by the theory of integrable systems, one usually restricts the class of PSS equations [4, 15]: intuitively, one would like to consider only PSS equations which are *necessary and sufficient* conditions for equations (2) to hold. In the case of evolution equations $u_t = F(x, t, u, \ldots, u_{x^k})$, one proceeds rigorously thus [8, 6].

Let I_F be the differential ideal generated by the two-forms

$$\mathrm{d}u \wedge \mathrm{d}x + F(x, t, u, \ldots, u_{x^k})\,\mathrm{d}x \wedge \mathrm{d}t \qquad \mathrm{d}u_{x^l} \wedge \mathrm{d}t - u_{x^{l+1}}\,\mathrm{d}x \wedge \mathrm{d}t$$

$1 \leqslant l \leqslant k - 1$, on a manifold J with coordinates $x, t, u, u_x, \ldots, u_{x^k}$, so that local solutions of $u_t = F$ correspond to integral sub-manifolds of the exterior differential system $\{I_F, \mathrm{d}x \wedge \mathrm{d}t\}$.

Definition 3. *An evolutionary equation* $u_t = F(x, t, u, \ldots, u_{x^k})$ *is strictly pseudo-spherical if there exist one-forms* $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ *whose coefficients* $f_{\alpha\beta}$ *are smooth functions on* J*, such that the two-forms*

$$\Omega_1 = \mathrm{d}\omega^1 - \omega^3 \wedge \omega^2 \qquad \Omega_2 = \mathrm{d}\omega^2 - \omega^1 \wedge \omega^3 \qquad \Omega_3 = \mathrm{d}\omega^3 - \omega^1 \wedge \omega^2 \tag{16}$$

generate I_F.

The linear equation $\widehat{u}_{\widehat{t}} = \widehat{u}_{\widehat{x}\widehat{x}} + \widehat{u}_{\widehat{x}}$ is an example of such an equation. Definition 3 implies that functions $f_{\alpha\beta}$ associated with strictly pseudo-spherical equations are constrained from the outset [12]:

Lemma 2. *Necessary and sufficient conditions for the kth order equation* $u_t = F$ *to be strictly pseudo-spherical are the conjunction of: (a) the functions* $f_{\alpha\beta}$ *satisfy* $f_{\alpha 1, u_{x^a}} = 0$; $f_{\alpha 2, u_{x^k}} = 0$; $f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0$*, in which* $a \geqslant 1$*; and (b) F and* $f_{\alpha\beta}$ *satisfy the identities*

$$-f_{\alpha 1,u} F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{\alpha 2, u_{x^i}} + f_{\delta 1} f_{\gamma 2} - f_{\gamma 1} f_{\delta 2} + f_{\alpha 2,x} - f_{\alpha 1,t} = 0 \tag{17}$$

in which (α, δ, γ) *is* $(1, 2, 3), (2, 3, 1), (3, 2, 1)$.

Definition 4. *A differential function G is a generalized symmetry of an evolution equation* $u_t = F$ *if for any local solution* $u(x, t)$ *of* $u_t = F$*, the function* $u(x, t) + \tau G(u(x, t))$ *satisfies the equation* $u_t = F$ *to first order in* τ.

Generalized symmetries of strictly pseudo-spherical equations are characterized as follows. Assume that $u_t = F$ is an mth order strictly pseudo-spherical equation with associated one-forms ω^α, and consider a local solution $u(x, t)$ of $u_t = F$. Set $\overline{G} = G(u(x, t))$, in which G is any differential function, and expand, using lemma 2, the one-forms $\omega^\alpha(u(x, t) + \tau\overline{G})$ about $\tau = 0$. One obtains an infinitesimal deformation $\overline{\varpi}^\alpha + \tau\overline{\Lambda}_\alpha$ of $\overline{\varpi}^\alpha = \omega^\alpha(u(x, t))$. One then proves ([13] and references therein):

Theorem 3. *Suppose that* $u_t = F(x, t, u, \ldots u_{x^m})$ *is strictly pseudo-spherical with associated one-forms* $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$. *Let G be a differential function, and consider the deformed one-forms* $\overline{\varpi}^\alpha + \tau\overline{\Lambda}_\alpha$ *defined above. They satisfy the structure equations of a pseudo-spherical surface up to terms of order* τ^2 *if and only if G is a generalized symmetry of* $u_t = F$.

Next, one introduces hierarchies of pseudo-spherical type [13], motivated by the fact that in the theory of integrable systems one is concerned not with single equations, but with countable numbers of equations which determine pairwise commuting flows. The independent variables will be $x, t, \tau_1, \tau_2, \ldots$.

Definition 5. *Let* $u_{\tau_i} = F_i, i \geqslant 0, \tau_0 = t$*, be a countable number of evolution equations. These equations form a hierarchy of equations describing pseudo-spherical surfaces (or a hierarchy of pseudo-spherical type) if there exist differential functions* $f_{\alpha\beta}$ *and* $h_{\alpha i}, i \geqslant 1$*, such that the one-forms*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t + \sum_{i=1}^{n} h_{\alpha i}\,\mathrm{d}\tau_i \tag{18}$$

satisfy the equations

$$d_H\Theta_1^{[n]} = \Theta_3^{[n]} \wedge \Theta_2^{[n]} \qquad d_H\Theta_2^{[n]} = \Theta_1^{[n]} \wedge \Theta_3^{[n]} \qquad d_H\Theta_3^{[n]} = \Theta_1^{[n]} \wedge \Theta_2^{[n]} \tag{19}$$

for all $n \geqslant 0$.

In (19), the exterior derivative $d_H \Theta_\alpha^{[n]}$ is computed by means of $d_H(\mathrm{d}x) = d_H(\mathrm{d}\tau_i) = 0$, and $d_H g = D_x g \,\mathrm{d}x + \sum_{i=0}^n D_{\tau_i} g \,\mathrm{d}\tau_i$ for any differential function g; the operators D_x, D_{τ_i}, $i \geqslant 0$ are given by

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} u_{x^{j+1}} \frac{\partial}{\partial u_{x^j}} \qquad \text{and} \qquad D_{\tau_i} = \frac{\partial}{\partial \tau_i} + \sum_{j=0}^{\infty} \left(D_x^j F_i\right) \frac{\partial}{\partial u_{x^j}}. \tag{20}$$

Example 3. The well-known KdV hierarchy is of pseudo-spherical type: one defines functions $f_{\alpha\beta}$ and $h_{\alpha i}$, $i \geqslant 0$, as follows: $f_{11} = 1 - u$, $f_{21} = \lambda$, $f_{31} = -1 - u$, $f_{12} = -u + 1$, $f_{22} = \lambda$, $f_{32} = -u - 1$, $h_{\alpha 0} = f_{\alpha 2}$, and for $i > 0$, $h_{1i} = (1/2)\lambda B_x^{(i)} - (1/2)B_{xx}^{(i)} - uB^{(i)} + B^{(i)}$; $h_{2i} = \lambda B^{(i)} - B_x^{(i)}$; and $h_{3i} = (1/2)\lambda B_x^{(i)} - (1/2)B_{xx}^{(i)} - uB^{(i)} - B^{(i)}$. Here, λ is a parameter, $B^{(i)} = \sum_{j=0}^{i} B_j \lambda^{2(i-j)}$, and the functions B_j, $j \geqslant 1$, are determined by the recursion relation

$$B_{0,x} = 0 \qquad B_{j+1,x} = B_{j,xxx} + 4uB_{j,x} + 2u_x B_j \quad 0 \leqslant j \leqslant n-1. \tag{21}$$

That (21) determine differential functions B_j is proven in [3]. The equation $u_{\tau_i} = F_i$ is the ith-order KdV equation, $u_{\tau_i} = (1/2)B_{n,xxx} + u_x B_n + 2uB_{n,x}$. For instance, one can easily check that the equation $u_{\tau_0} = F_0$ is the standard KdV equation $u_t = u_{xxx} + 6uu_x$.

Geometrically, equations (19) say that the one-forms $\Theta_\alpha^{[n]}$ describe pseudo-spherical surfaces immersed in a flat pseudo-Riemannian manifold of dimension $n + 2$. A rigorous theorem to this effect appears in [13]. The following result is also proven in [13]:

Theorem 4. *Let $u_{\tau_i} = F_i$, $i \geqslant 0$, be a hierarchy of pseudo-spherical type with associated one-forms $\Theta_\alpha^{[n]} = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t + \sum_{i=1}^n h_{\alpha i}\,\mathrm{d}\tau_i$, $n \geqslant 0$. Then, the equations $u_{\tau_i} = F_i$ describe pseudo-spherical surfaces with associated one-forms $\omega_0^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ if $i = 0$, and $\omega_i^\alpha = f_{\alpha 1}\,\mathrm{d}x + h_{\alpha i}\,\mathrm{d}\tau_i$, if $i \geqslant 1$. Moreover, the differential function F_j is a generalized symmetry of the equation $u_{\tau_i} = F_i$ for all $i, j \geqslant 0$.*

Finally, one can show that the transformations of section 2 can be generalized to the hierarchy case. Only a general version of theorem 1 will be presented here, a hierarchy version of theorem 2 is more involved and it appears in [13]:

Definition 6. *A solution of a hierarchy of pseudo-spherical type $u_{\tau_i} = F_i$, $i \geqslant 0$ is a sequence $\{u^{[n]}(x, t, \tau_1, \ldots, \tau_n)\}_{n \geqslant 0}$ of smooth functions $u^{[n]} : V^{[n]} \subset \mathbf{R}^{n+2} \to \mathbf{R}$ such that for each $n \geqslant 0$ one has: (a)$u^{[n]}$ is a solution of the equations $u_{\tau_i} = F_i$, $i = 1, \ldots, n$, and (b)$u^{[n+1]}|_{V^{[n]}} = u^{[n]}$.*

Theorem 5. *Let $u_{\tau_i} = F_i(x, t, u, \ldots)$ and $\widehat{u}_{\widehat{\tau}_i} = \widehat{F}_i(\widehat{x}, \widehat{t}, \widehat{u}, \ldots)$, $i \geqslant 0$ be two hierarchies of pseudo-spherical type with associated one-forms $\Theta_\alpha^{[n]}$ and $\widehat{\Theta}_\alpha^{[n]}$, respectively. Let $\{u^{[n]}\}$ and $\{\widehat{u}^{[n]}\}$ be solutions of $u_{\tau_i} = F_i$ and $\widehat{u}_{\widehat{\tau}_i} = \widehat{F}_i$, $i \geqslant 0$, and assume that $u^{(0)}(x, t)$ and $\widehat{u}^{(0)}(\widehat{x}, \widehat{t})$ are III-generic.*

For each $n \geqslant 0$ there exists a local diffeomorphism $\Upsilon^{[n]} : V^{[n]} \to \widehat{V}^{[n]}$, in which $V^{[n]}$ and $\widehat{V}^{[n]}$ are open subsets of the domains of $u^{[n]}$ and $\widehat{u}^{[n]}$, respectively, and a smooth function $\mu^{[n]} : V^{[n]} \to \mathbf{R}$, such that the pull-backs of $\Theta_\alpha^{[n]}$ by $u^{[n]}$ and $\widehat{\Theta}_\alpha^{[n]}$ by $\widehat{u}^{[n]}$ satisfy

$$\Upsilon^{[n]*}\widehat{\Theta}_1^{[n]} = \Theta_1^{[n]} \cos\mu^{[n]} + \Theta_2^{[n]} \sin\mu^{[n]} \tag{22}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_2^{[n]} = -\Theta_1^{[n]} \sin\mu^{[n]} + \Theta_2^{[n]} \cos\mu^{[n]} \tag{23}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_3^{[n]} = \Theta_3^{[n]} + \mathrm{d}\mu^{[n]}. \tag{24}$$

Moreover, the maps $\Upsilon^{[n]}$ and $\mu^{[n]}$, $n \geqslant 0$ can be chosen so that

$$\Upsilon^{[n+1]}|_{V^{[n]}} = \Upsilon^{[n]} \qquad \textit{and} \qquad \mu^{[n+1]}|_{V^{[n]}} = \mu^{[n]} \qquad n \geqslant 0. \tag{25}$$

Example 4. The hierarchy $\widehat{u}_{\widehat{\tau}_i} = \widehat{F}_i$, $i \geqslant 0$, with $\widehat{u}_{\widehat{t}} = \widehat{u}_{\widehat{x}\widehat{x}} + \widehat{u}_{\widehat{x}}$, and

$$\widehat{u}_{\widehat{\tau}_i} = a_{i+1}^{i+2}\widehat{u}_{\widehat{x}^{i+2}} + \sum_{l=1}^{i} a_l^{i+2}\widehat{u}_{\widehat{x}^{l+1}} + \sum_{l=1}^{i+1} a_l^{i+2}\widehat{u}_{\widehat{x}^{l}} \tag{26}$$

in which the constants a_s^r are arbitrary except that $a_1^r = 1$, $r \geqslant 1$ is of pseudo-spherical type with associated functions $\widehat{f}_{11} = \widehat{u}$, $\widehat{f}_{21} = 1$, $\widehat{f}_{31} = \widehat{u}$, $\widehat{h}_{1i} = \sum_{k=1}^{i+1} a_k^{i+2}\widehat{u}_{\widehat{x}^k}$, $\widehat{h}_{2i} = 0$, $\widehat{h}_{3i} = \widehat{h}_{1i}$ and $f_{\alpha 2} = h_{\alpha 0}$.

For a given hierarchy $u_{\tau_i} = F_i$ with associated one-forms (18), one can connect its solutions $\{u^{[n]}\}_{n\geqslant 0}$ such that $u^{[0]}(x,t)$ is *III*-generic and the solution $\widehat{u}^{[n]} = \widehat{x} + \widehat{t} + \widehat{\tau}_1 + \cdots + \widehat{\tau}_n$ to (26) as follows: because of (19), one can find solutions $\alpha(x,t,\ldots,\tau_n)$, $\beta(x,t,\ldots,\tau_n)$ and $\theta^{[n]}(x,t,\ldots,\tau_n)$ to the equations $\Theta_3^{[n]} + \mathrm{d}\theta^{[n]} = \Theta_1^{[n]}\cos\theta^{[n]} + \Theta_2^{[n]}\sin\theta^{[n]}$, $\mathrm{d}(\ln\beta) = -\Theta_1^{[n]}\sin\theta^{[n]} + \Theta_2^{[n]}\cos\theta^{[n]}$, and $\mathrm{d}\big(\alpha + \sum_{i=1}^n \tau_i\big) = \beta\big(\Theta_1^{[n]}\cos\theta^{[n]} + \Theta_2^{[n]}\sin\theta^{[n]}\big)$, and (see [13]) one can assume that $\theta^{[n+1]}|_{V^{[n]}} = \theta^{[n]}$. (Notation as in theorem 5.) Define $\Upsilon^{[n]} : (x,t,\tau_1,\ldots,\tau_n) \mapsto (\widehat{x},\widehat{t},\widehat{\tau}_1,\ldots,\widehat{\tau}_n)$ by

$$\widehat{x} = -\ln\left|\beta + \frac{1}{\beta}\left(\alpha + \sum_{i=1}^{n}\tau_i\right)^2\right| \tag{27}$$

$$\widehat{t} = -\left(\frac{\alpha + \sum_{i=1}^{n}\tau_i}{\beta}\right) + 1 - \sum_{i=1}^{n}\tau_i + \ln\left|\beta + \frac{1}{\beta}\left(\alpha + \sum_{i=1}^{n}\tau_i\right)^2\right| \tag{28}$$

$$\widehat{\tau}_i = \tau_i \tag{29}$$

and set $\mu^{[n]} = \theta^{[n]} - \widehat{\theta}^{[n]} \circ \Upsilon^{[n]}$, in which $\widehat{\theta}^{[n]}$ is determined by

$$\cos\widehat{\theta}^{[n]} = \frac{-1 + (\widehat{u}^{[n]} - 1)^2}{1 + (\widehat{u}^{[n]} - 1)^2} \qquad \sin\widehat{\theta}^{[n]} = \frac{2(\widehat{u}^{[n]} - 1)}{1 + (\widehat{u}^{[n]} - 1)^2}.$$

These maps satisfy the conclusions of theorem 5.

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